CS 6630 Fuzzy Logic Midterm Solution

1.

a) let $C_1 = A \cap \overline{B}$, $\mu_{C_1} = \mu_{A\cap\overline{B}} = \min(\mu_A, \mu_{\overline{B}}) = \min(\mu_A, 1-\mu_B)$,

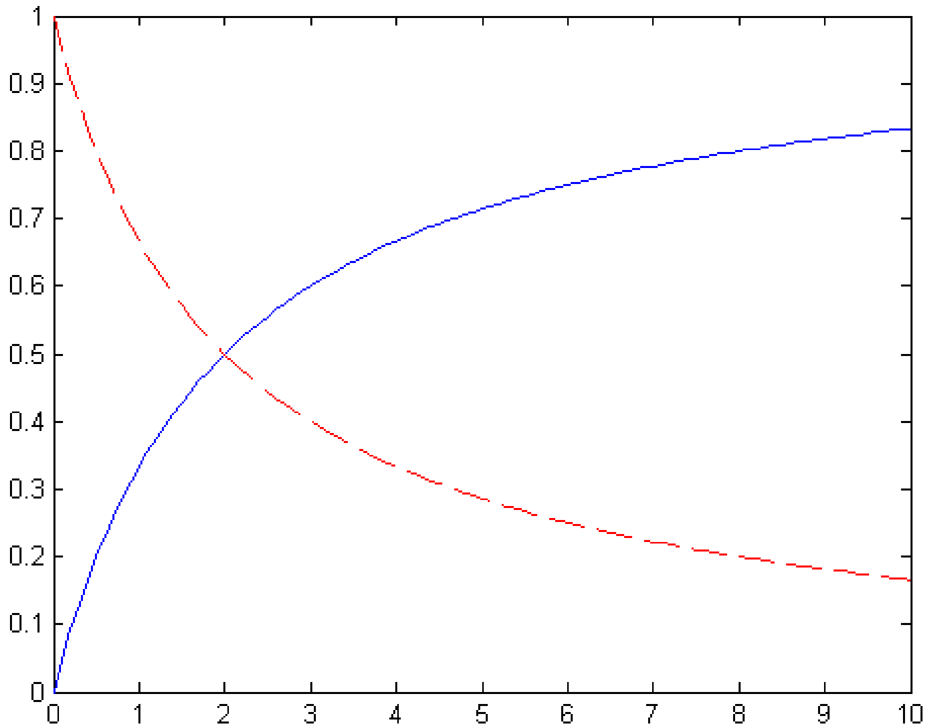
$$\frac{1-\mu_B}{\mu_A} = \frac{1-2^{-x}}{\frac{x}{x+2}} = \frac{x+2}{x} - \frac{x+2}{2^x x} = 1 + \frac{2^{x+1}-(x+2)}{2^x x} \qquad x \in [0,10] \text{, here}$$

we have $2^{x+1} = (1+1)^{x+1} \geq 1 + x + 1 = x+2$, so

$\frac{1-\mu_B}{\mu_A} \geq 1 + \frac{(x+2)-(x+2)}{2^x x} = 1$, i.e., $1-\mu_B \geq \mu_A$, therefore,

$\mu_{C_1} = \min(\mu_A, 1-\mu_B) = \mu_A$

b) Let $C_2 = \overline{A \cap \overline{B}}$, i.e., $C_2 = \overline{C_1}$, therefore, $\mu_{C_2} = 1 - \mu_{C_1} = 1 - \mu_A = \mu_{\overline{A}}$



μ_{C_1} is the blue solid line and μ_{C_2} is the red dash line

2.

We prove the follow formula first

a) For any fuzzy sets x and y, $x \cap \overline{x} \subseteq (x \cap y) \cup (\overline{x} \cap \overline{y})$. It is simple, if the sets are crisp sets, that $x \cap \overline{x} = \Phi$. For the fuzzy sets, however, it is not trivial.

Proof:

i) If $\mu_x \leq 1 - \mu_y$,

$$\mu_{x\cap\overline{x}} = \min(\mu_x, 1-\mu_x) \leq \min(1-\mu_y, 1-\mu_x) = \mu_{\overline{x}\cap\overline{y}} \Rightarrow x \cap \overline{x} \subseteq \overline{x} \cap \overline{y}$$

ii) Otherwise, $\mu_x > 1-\mu_y \Rightarrow \mu_y > 1-\mu_x$,

$$\mu_{x\cap\bar{x}} = \min(\mu_x, 1-\mu_x) \le \min(\mu_x, \mu_y) = \mu_{x\cap y} \Rightarrow x\cap\bar{x} \subseteq x\cap y$$

Therefore, $x\cap\bar{x} \subseteq (x\cap y)\cup(\bar{x}\cap\bar{y})$

b)

$$(A\Delta B)\Delta C = [(A-B)\cup(B-A)]\Delta C$$
$$= \{[(A-B)\cup(B-A)]-C\}\cup\{C-[(A-B)\cup(B-A)]\}$$
$$= \{[(A\cap\overline{B})\cup(B\cap\overline{A})]\cap\overline{C}\}\cup\{C\cap[\overline{(A\cap\overline{B})\cup(B\cap\overline{A})}]\}$$
$$= [(A\cap\overline{B}\cap\overline{C})\cup(B\cap\overline{A}\cap\overline{C})]\cup\{C\cap[(\overline{A}\cup B)\cap(\overline{B}\cup A)]\}$$
$$= [(A\cap\overline{B}\cap\overline{C})\cup(B\cap\overline{A}\cap\overline{C})]\cup\{C\cap[(\overline{A}\cap\overline{B})\cup(\overline{A}\cap A)\cup(B\cap\overline{B})\cup(B\cap A)]\}$$

according to a), $\overline{A}\cap A,\quad B\cap\overline{B} \quad \subseteq (\overline{A}\cap\overline{B})\cup(B\cap A)$

$$= [(A\cap\overline{B}\cap\overline{C})\cup(B\cap\overline{A}\cap\overline{C})]\cup\{C\cap[(\overline{A}\cap\overline{B})\cup(B\cap A)]\}$$
$$= (A\cap\overline{B}\cap\overline{C})\cup(\overline{A}\cap B\cap\overline{C})\cup(\overline{A}\cap\overline{B}\cap C)\cup(A\cap B\cap C)$$

$$A\Delta(B\Delta C) = (A-B\Delta C)\cup(B\Delta C-A)$$
$$= (A\cap\overline{B\Delta C})\cup(B\Delta C\cap A)$$
$$= [A\cap(\overline{B-C}\cap\overline{C-B})]\cup\{[(B-C)\cup(C-B)]\cap A\}$$
$$= \{A\cap[(\overline{B}\cup C)\cap(\overline{C}\cup B)]\}\cup\{[(B\cap\overline{C})\cup(C\cap\overline{B})]\cap A\}$$
$$= \{A\cap[(\overline{B}\cap B)\cup(\overline{B}\cap\overline{C})\cup(C\cap\overline{C})\cup(C\cap B)]\}\cup(A\cap B\cap\overline{C})\cup(A\cap\overline{B}\cap C)$$

according to a), $\overline{B}\cap B,\quad C\cap\overline{C} \quad \subseteq(\overline{B}\cap\overline{C})\cup(C\cap B)$

$$= \{A\cap[(\overline{B}\cap\overline{C})\cup(C\cap B)]\}\cup(A\cap B\cap\overline{C})\cup(A\cap\overline{B}\cap C)$$
$$= (A\cap\overline{B}\cap\overline{C})\cup(\overline{A}\cap B\cap\overline{C})\cup(\overline{A}\cap\overline{B}\cap C)\cup(A\cap B\cap C)$$

$\therefore \quad RHS = LHS$ #

3.

Final report is either research-like or survey-like. Both should follow the style of published paper, however, they DO NOT need to be published.

Research-like: (need idea and better result)

i) Title page: title, name, and abstract
ii) Introduction
iii) Proposed method
iv) Experiments
v) Conclusions
vi) References

Survey-like: (need read at least 30 papers)

i) Title page: title, name, and abstract

ii) Introduction
iii) Comparing different methods, giving the advantages and disadvantages
iv) Conclusions
v) References

Submit all the cited references with the final report together. For the book reference, submit the copies of the related pages only.

4.
See the solution for the homework #2 problem #5

5.

Commutativity:
$$xt_3y = 1 - \min(1, \sqrt[p]{(1-x)^p + (1-y)^p}) = 1 - \min(1, \sqrt[p]{(1-y)^p + (1-x)^p}) = yt_3x$$

Associativity:
i)
$$(xt_3y)t_3z = 1 - \min(1, \sqrt[p]{\{1 - [1 - \min(1, \sqrt[p]{(1-x)^p + (1-y)^p})]\}^p + (1-z)^p})$$
$$= 1 - \min(1, \sqrt[p]{(\min(1, \sqrt[p]{(1-x)^p + (1-y)^p}))^p + (1-z)^p})$$
$$= 1 - \min(1, \sqrt[p]{1 + (1-z)^p}, \sqrt[p]{(1-x)^p + (1-y)^p + (1-z)^p})$$
$$= 1 - \min(1, \sqrt[p]{(1-x)^p + (1-y)^p + (1-z)^p})$$
$$Note \quad \sqrt[p]{1 + (1-z)^p} \geq 1$$
ii)
$$xt_3(yt_3z) = 1 - \min(1, \sqrt[p]{(1-x)^p + \{1 - [1 - \min(1, \sqrt[p]{(1-y)^p + (1-z)^p})]\}^p})$$
$$= 1 - \min(1, \sqrt[p]{(1-x)^p + (\min(1, \sqrt[p]{(1-y)^p + (1-x)^p}))^p})$$
$$= 1 - \min(1, \sqrt[p]{1 + (1-x)^p}, \sqrt[p]{(1-x)^p + (1-y)^p + (1-z)^p})$$
$$= 1 - \min(1, \sqrt[p]{(1-x)^p + (1-y)^p + (1-z)^p})$$
$$Note \quad \sqrt[p]{1 + (1-x)^p} \geq 1$$

From i) and ii), we conclude $(xt_3y)t_3z = xt_3(yt_3z)$

Monotonicity:
If $x \leq y$ and $w \leq z$, then
$$\sqrt[p]{(1-x)^p + (1-w)^p} \geq \sqrt[p]{(1-y)^p + (1-z)^p}$$
$$\Rightarrow \min(1, \sqrt[p]{(1-x)^p + (1-w)^p}) \geq \min(1, \sqrt[p]{(1-y)^p + (1-z)^p})$$
$$xt_3w = 1 - \min(1, \sqrt[p]{(1-x)^p + (1-w)^p}) \leq 1 - \min(1, \sqrt[p]{(1-y)^p + (1-z)^p}) = yt_3z$$

Boundary conditions:

$0t_3x = 1 - \min(1, \sqrt[p]{1+(1-x)^p}) = 1 - 1 = 0$

Note $\sqrt[p]{1+(1-x)^p} \geq 1$

$1t_3x = 1 - \min(1, \sqrt[p]{0+(1-x)^p}) = 1-(1-x) = x$

6.

Here we only prove the associative law

$(xs_2y)s_2z = \min(1, x+y+pxy)s_2z$

$$= \begin{cases} 1s_2z & if \quad x+y+pxy \geq 1 \\ (x+y+pxy)s_2z & otherwise \end{cases}$$

$$= \begin{cases} 1 & if \quad x+y+pxy \geq 1 \\ \min(1, x+y+z+pxy+pyz+pxz+p^2xyz) & otherwise \end{cases}$$

Let $\beta = x+y+z+pxy+pyz+pxz+p^2xyz$, then

$$(xs_2y)s_2z = \begin{cases} 1 & otherwise \\ \beta & \beta < 1 \end{cases}$$

$xs_2(ys_2z) = xs_2(\min(1, y+z+pyz))$

$$= \begin{cases} xs_21 & if \quad y+z+pyz \geq 1 \\ xs_2(y+z+pyz) & otherwise \end{cases}$$

$$= \begin{cases} 1 & if \quad y+z+pyz \geq 1 \\ \min(1, x+y+z+pxy+pyz+pxz+p^2xyz) & otherwise \end{cases}$$

That is,

$$xs_2(ys_2z) = \begin{cases} 1 & otherwise \\ \beta & \beta < 1 \end{cases}$$

$(xs_2y)s_2z = xs_2(ys_2z)$ and the associative law is satisfied